Qualitative analysis of string cosmologies

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A qualitative analysis is presented for spatially flat, isotropic and homogeneous cosmologies derived from the string effective action when the combined effects of a dilaton, modulus, two-form potential and central charge deficit are included. The latter has significant effects on the qualitative dynamics. The analysis is also directly applicable to the anisotropic Bianchi type I cosmology. [S0556-2821(99)01112-1]

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I. INTRODUCTION

There has been considerable interest recently in the cosmological implications of string theory. String theory introduces significant modifications to the standard, hot big bang model based on conventional Einstein gravity and early universe cosmology provides one of the few environments where the predictions of the theory can be quantitatively investigated. A study of string cosmologies is therefore well motivated.

The evolution of the very early universe much below the string scale and for string coupling much smaller than unity, $g_s \ll 1$, is determined by ten-dimensional supergravity theories [1–3]. All theories of this type contain a dilaton, a graviton and a two-form potential in the Neveu-Schwarz–Neveu-Schwarz (NS-NS) bosonic sector. If one considers a Kaluza-Klein compactification from ten dimensions onto an isotropic six-torus of radius e^{β}, the effective action is given by

$$S=\int d^4x\sqrt{-g}e^{-\Phi}\Big[R+(\nabla\Phi)^2-6(\nabla\beta)^2-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}-2\Lambda\Big], \tag{1.1}$$

where R is the Ricci curvature of the spacetime with metric $g_{\mu\nu}$, $g\equiv\det g_{\mu\nu}$, the dilaton field, Φ, parametrizes the string coupling, $g_s^2\equiv e^{\Phi}$, and $H_{\mu\nu\lambda}\equiv\partial_{[\mu}B_{\nu\lambda]}$ is the field strength of the two-form potential, $B_{\mu\nu}$. The volume of the internal dimensions is parametrized by the modulus field, β. The moduli fields arising from the compactification of the two-form on the internal dimensions have been neglected [4]. The constant, Λ, is determined by the central charge deficit of the string theory. In principle, it may take arbitrary values if the string is coupled to an appropriate conformal field theory. Such a term may also have an origin in terms of the reduction of higher degree form fields [5].

In four dimensions, the three-form field strength is dual to a one-form:

$$H^{\mu\nu\lambda}\equiv e^{\Phi}\epsilon^{\mu\nu\lambda\kappa}\nabla_{\kappa}\sigma, \tag{1.2}$$

where $\epsilon^{\mu\nu\lambda\kappa}$ is the covariantly constant four-form. In this dual formulation, the field equations can be derived from the action

$$S=\int d^4x\sqrt{-g}e^{-\Phi}\Big[R+(\nabla\Phi)^2-6(\nabla\beta)^2-\frac{1}{2}e^{2\Phi}(\nabla\sigma)^2-2\Lambda\Big], \tag{1.3}$$

where σ is interpreted as a pseudo-scalar ''axion'' field [6]. It can be shown that the action (1.3) is invariant under a global SL(2,R) transformation on the dilaton and axion fields when Λ vanishes [6]. The general Friedmann-Robertson-Walker (FRW) cosmologies derived from Eq. (1.3) with $\Lambda=0$ have been found by employing this symmetry [7]. However, the symmetry is broken when a stringy cosmological constant is present [8] and the general FRW solution is not known in this case.

The purpose of the present paper is to determine the general structure of the phase space of solutions for the class of spatially flat, FRW string cosmologies derived from the effective action (1.3) when a cosmological constant is present. This is well motivated from a theoretical point of view and is also relevant in light of recent high redshift observations that indicate a vacuum energy density may be dominating the large-scale dynamics of the universe at the present epoch [9].

The paper is organized as follows. In Sec. II, the field equations are presented as an autonomous system of ordinary differential equations (ODEs). The combined effects of the axion, modulus and dilaton fields are determined for a negative and positive central charge deficit in Secs. III and IV, respectively. This extends previous qualitative analyses where one or more of these terms was neglected [10–14]. A full stability analysis is performed for all cases by rewriting the field equations in terms of a set of compactified variables. We conclude in Sec. V with a discussion of the phase portraits.

II. COSMOLOGICAL FIELD EQUATIONS

The spatially flat, FRW cosmological field equations derived from the action (1.3) are given by

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$$2\ddot{\alpha}-2\dot{\alpha}\dot{\varphi}-\dot{\sigma}^2e^{2\varphi+6\alpha}=0 \tag{2.1}$$

$$2\ddot{\varphi}-\dot{\varphi}^2-3\dot{\alpha}^2-6\dot{\beta}^2+\frac{1}{2}\dot{\sigma}^2e^{2\varphi+6\alpha}+2\Lambda=0 \tag{2.2}$$

$$\ddot{\beta}-\dot{\beta}\dot{\varphi}=0 \tag{2.3}$$

$$\ddot{\sigma}+\dot{\sigma}(\dot{\varphi}+6\dot{\alpha})=0, \tag{2.4}$$

where

$$\varphi\equiv\Phi-3\alpha \tag{2.5}$$

defines the ''shifted'' dilaton field, $a\equiv e^{\alpha}$ is the scale factor of the universe and an overdot denotes differentiation with respect to cosmic time, t. The generalized Friedmann constraint equation is

$$3\dot{\alpha}^2-\dot{\varphi}^2+6\dot{\beta}^2+\frac{1}{2}\dot{\sigma}^2e^{2\varphi+6\alpha}+2\Lambda=0. \tag{2.6}$$

A number of exact solutions to Eqs. (2.1)–(2.6) are known when one or more of the degrees of freedom are trivial. We now discuss those that represent the invariant sets of the full phase space of solutions. The ''dilaton-vacuum'' solutions, where only the dilaton field is dynamically important, are given by

$$a=a_*|t|^{1/p_\pm} \tag{2.7}$$

$$e^{\Phi}=e^{\Phi_*}|t|^{p_\pm-1},$$

where $p_\pm\equiv\pm\sqrt{3}$ and $\{a_*,\Phi_*\}$ are arbitrary constants. There is a curvature singularity at $t=0$. The solution (2.7) forms the basis of the pre-big-bang scenario, where a growing string coupling can drive an epoch of inflationary (accelerated) expansion [15]. The pre-big-bang phase corresponds to the $p=p_-$ solution over the range $t<0$ and the post-big-bang phase to the $p=p_+$ solution for $t>0$ [15]. The inflationary nature of this scenario has recently been questioned, however [16].

The ''dilaton-moduli-vacuum'' solutions, with $\dot{\sigma}=\Lambda=0$, are given by

$$a=a_*|t|^{\pm h_*} \tag{2.8}$$

$$e^{\Phi}=e^{\Phi_*}|t|^{\pm 3h_*-1}$$

$$e^{\beta}=e^{\beta_*}|t|^{\pm\sqrt{(1-3h_*^2)/6}},$$

where $\{h_*,\beta_*\}$ are constants. This class of solution can also be expressed in the form

$$a=a_*\left|\frac{s}{s_*}\right|^{(1+r_\pm)/2} \tag{2.9}$$

$$e^{\Phi}=e^{\Phi_*}\left|\frac{s}{s_*}\right|^{r_\pm}$$

$$e^{\beta}=e^{\beta_*}\left|\frac{s}{s_*}\right|^{q},$$

where $\{s_*,q,r_\pm\}$ are constants and $s\equiv\int^t dt'/a(t')$ is conformal time. The generalized Friedmann constraint equation (2.6) leads to the constraint $r_\pm=\pm(3-12q^2)^{1/2}$.

The general solution where only $\Lambda=0$ is the ''dilaton-moduli-axion'' solution [7]

$$a=a_*\left|\frac{s}{s_*}\right|^{1/2}\left[\left|\frac{s}{s_*}\right|^{r}+\left|\frac{s}{s_*}\right|^{-r}\right]^{1/2}$$

$$e^{\Phi}=\frac{e^{\Phi_*}}{2}\left[\left|\frac{s}{s_*}\right|^{r}+\left|\frac{s}{s_*}\right|^{-r}\right]$$

$$\sigma=\sigma_*\pm e^{-\Phi_*}\left[\frac{|s/s_*|^{-r}-|s/s_*|^{r}}{|s/s_*|^{-r}+|s/s_*|^{r}}\right]$$

$$e^{\beta}=e^{\beta_*}\left|\frac{s}{s_*}\right|^{q}, \tag{2.10}$$

where σ_* is an arbitrary constant and $r\equiv|r_\pm|$. This cosmology asymptotically approaches one of the dilaton-moduli-vacuum solutions (2.9) in the limits of high and low spacetime curvature. The axion field induces a smooth transition between these two power-law solutions and causes a bounce to occur. It is only dynamically important for a short time interval when $s\approx s_*$ [7].

The solutions where only the axion field is trivial and $\Lambda>0$ are specific cases of the ''rolling radii'' solutions [17]

$$a=a_*|\tanh(At/2)|^{m}$$

$$e^{-\Phi}=e^{-\Phi_*}|\cosh(At/2)|^{2k-6n}|\sinh(At/2)|^{2l+6n}$$

$$e^{\beta}=e^{\beta_*}|\tanh(At/2)|^{n}, \tag{2.11}$$

where $A\equiv\sqrt{2\Lambda}$ and the real numbers $\{k,l,m,n\}$ satisfy the constraints

$$3m^2+6n^2=1,\quad 3m+6n=k-l,\quad k+l=1. \tag{2.12}$$

The corresponding solutions for $\Lambda<0$ are related to Eqs. (2.11) by redefining $A\equiv-i\tilde{A}$. In this case, the range of t is bounded such that $0<t<\pi/\tilde{A}$.

Finally, there exists the ''linear dilaton vacuum'' solution where $\Lambda>0$ [18]. This solution is static and the dilaton evolves linearly with time:

$$\dot{a}=0,\quad \dot{\beta}=0,\quad \Phi=\pm\sqrt{2\Lambda}t. \tag{2.13}$$

The field equations (2.1)–(2.6) may be written as the following system of autonomous ODEs:

$$\dot{h}=\psi^2+h\psi-3h^2-N-2\Lambda \tag{2.14}$$

$$\dot{\psi}=3h^2+N \tag{2.15}$$

$$\dot{N}=2N\psi \tag{2.16}$$

$$\dot{\rho} = -6h\rho \tag{2.17}$$

$$3h^2 - \psi^2 + N + \frac{1}{2}\rho + 2\Lambda = 0, \tag{2.18}$$

where we have defined the new variables

$$N \equiv 6\dot{\beta}^2, \quad \rho \equiv \dot{\sigma}^2 e^{2\varphi + 6\alpha}, \quad \psi \equiv \dot{\varphi}, \quad h \equiv \dot{\alpha}. \tag{2.19}$$

The variable ρ may be interpreted as the effective energy density of the pseudo-scalar axion field [13]. It follows from Eq. (2.15) that ψ is a monotonically increasing function of time and this implies that the equilibrium points of the system of ODEs must be located either at zero or infinite values of ψ. In addition, as a result of the existence of a monotone function, it follows that there are no periodic or recurrent orbits in the phase space [19,20]. The sets $\Lambda = 0$ and $\rho = 0$ are invariant sets. In particular, the exact solution for $\Lambda = 0$ given by Eqs. (2.10) divides the phase space and the orbits do not cross from positive to negative Λ.

We now proceed to consider the cases where $\Lambda < 0$ and $\Lambda > 0$ separately.

III. ANALYSIS FOR NEGATIVE CENTRAL CHARGE DEFICIT

A. Four-dimensional model

In this section we consider the phase portraits of the NS-NS fields for negative central charge deficit. In the case where the modulus field is frozen, $N = 0$, Eq. (2.18) may be employed to eliminate the axion field's energy density. This reduces the set of equations (2.14)–(2.17) to a two-dimensional system. Moreover, it follows from Eq. (2.18) that

$$\psi^2 - 2\Lambda \geqslant 3h^2 \geqslant 0 \tag{3.1}$$

and we may therefore compactify the phase space by defining the new variables

$$\eta \equiv \frac{\psi}{\sqrt{\psi^2 - 2\Lambda}} \tag{3.2}$$

$$\xi \equiv \frac{\sqrt{3}h}{\sqrt{\psi^2 - 2\Lambda}} \tag{3.3}$$

and a new time variable

$$\frac{d}{d\tau} \equiv \frac{1}{\sqrt{\psi^2 - 2\Lambda}} \frac{d}{dt}. \tag{3.4}$$

Equations (2.14) and (2.15) then become

$$\frac{d\eta}{d\tau} = \xi^2(1 - \eta^2) \tag{3.5}$$

$$\frac{d\xi}{d\tau} = (\sqrt{3} + \eta\xi)(1 - \xi^2). \tag{3.6}$$

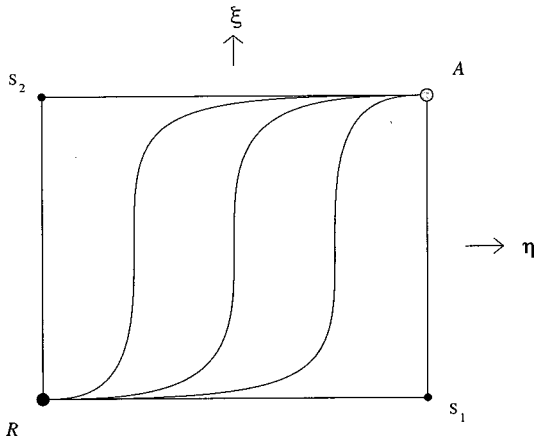

FIG. 1. The phase portrait of the system (3.5),(3.6) corresponding to the four-dimensional NS-NS model with no modulus field and negative central charge deficit ($\Lambda < 0$). Equilibrium points are denoted by dots and the labels in all figures correspond to those equilibrium points (and hence the exact solutions they represent) discussed in the text. We shall adopt the convention throughout that large black dots represent sources (i.e., repellers), large grey-filled dots represent sinks (i.e., attractors), and small black dots represent saddles. Arrows on the trajectories have been suppressed since the direction of increasing time is clear using this notation.

The variables defined in Eqs. (3.2) and (3.3) are bounded, $\eta^2 \leqslant 1$ and $\xi^2 \leqslant 1$, and it follows from Eqs. (3.5) and (3.6) that they are both monotonically increasing functions. The equilibrium points are located at $\xi^2 = \eta^2 = 1$. The invariant sets $\rho = 0$ and $\Lambda = 0$ correspond to the conditions $\xi^2 = 1$ and $\eta^2 = 1$, respectively. A stability analysis indicates that the equilibrium point $A{:}(\eta, \xi) = (1,1)$ is an attractor and the point $R{:}(\eta, \xi) = (-1,-1)$ is a repeller. The points $S_{1,2}{:}(1, -1)$ and $(-1,1)$ are both saddles. The phase portrait is given in Fig. 1 and is discussed in Sec. V, where a physical interpretation is given.

B. Ten-dimensional model

We now consider the effect of lifting the solutions to ten dimensions by including the modulus field, β. This will raise the dimension of the phase space to 3. In this case, it proves convenient to employ the generalized Friedmann constraint equation (2.18) to eliminate the modulus field rather than the axion field. This equation can be written as

$$1 - \xi^2 - \kappa = \frac{N}{\psi^2 - 2\Lambda}, \tag{3.7}$$

where the new variable κ is defined by

$$\kappa \equiv \frac{\rho}{2(\psi^2 - 2\Lambda)} \tag{3.8}$$

and satisfies $0 \leqslant \kappa \leqslant 1$. Employing Eqs. (3.2)–(3.4), we can now express the field equations (2.14)–(2.17) in the form

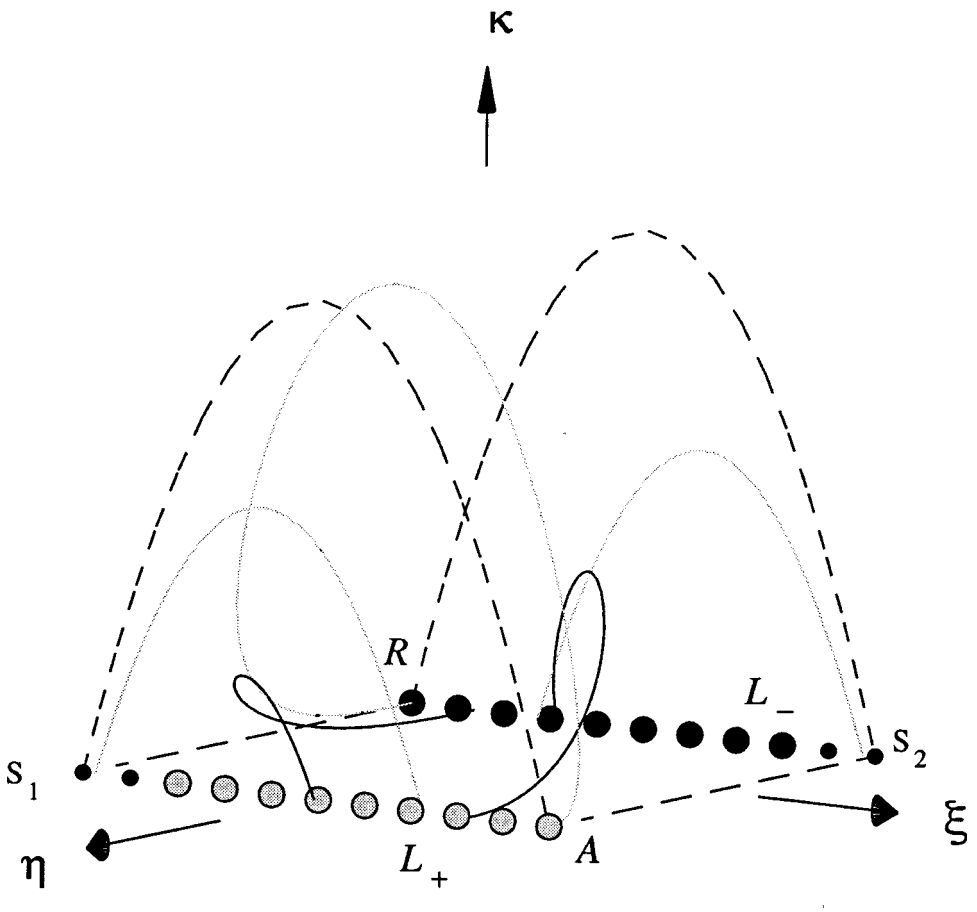

FIG. 2. The phase portrait of the system (3.9)–(3.11). This corresponds to the ten-dimensional NS-NS model with negative central charge deficit ($\Lambda<0$). Grey lines represent typical trajectories found within the two-dimensional invariant sets, dashed black lines are those trajectories along the intersection of the invariant sets, and solid black lines are typical trajectories within the full three-dimensional phase space. Note that $L_{\pm}$ denote lines of non-isolated equilibrium points. See also caption to Fig. 1.

$$\frac{d\eta}{d\tau}=(1-\kappa)(1-\eta^2) \tag{3.9}$$

$$\frac{d\xi}{d\tau}=\kappa(\sqrt{3}+\eta\xi) \tag{3.10}$$

$$\frac{d\kappa}{d\tau}=-2\kappa[\sqrt{3}\xi+\eta(1-\kappa)]. \tag{3.11}$$

We note that the invariant set $N=0$ corresponds to $\kappa=1-\xi^2$, in which case the above system of ODEs reduces to the two-dimensional system (3.5),(3.6).

The equilibrium points of this system of ODEs all lie on one of the two lines of non-isolated equilibrium points (or one-dimensional equilibrium sets)

$$L_{\pm}:\eta_0^2=1,\kappa=0, \tag{3.12}$$

where ξ is arbitrary. The corresponding eigenvalues are $\lambda_1=-2\eta_0$ and $\lambda_2=-2(\sqrt{3}\xi+\eta_0)$, and hence these equilibrium sets are normally hyperbolic (throughout, we shall refrain from giving the corresponding eigenvectors explicitly). The third eigenvalue is zero since this is a set of equilibrium points. Thus, on the line $L_+:(\eta_0=1,\kappa=0;\xi)$ the equilibrium points are saddles for $\xi\in[-1,-1/\sqrt{3})$ and local sinks for $\xi\in(-1/\sqrt{3},1]$. On the line $L_-:(\eta_0=-1,\kappa=0;\xi)$ the equilibrium points are local sources for $\xi\in[-1,1/\sqrt{3})$ and saddles for $\xi\in(1/\sqrt{3},1]$. The phase portrait is given in Fig. 2. The dynamics is very simple due to the fact that the right-hand sides of Eqs. (3.9) and (3.10) are positive-definite and hence η and ξ are always monotonically increasing functions. The curved upper boundary $\kappa=1-\xi^2$ denotes the invariant set $N=0$ and therefore corresponds to Fig. 1.

IV. ANALYSIS FOR POSITIVE CENTRAL CHARGE DEFICIT

In the case where the central charge deficit is positive, the variable $\psi^2-2\Lambda$ is no longer positive-definite and therefore cannot be employed to normalize the system. In view of this, we choose the normalization

$$\epsilon\equiv\left(3h^2+\frac{1}{2}\rho+N+2\Lambda\right)^{1/2}. \tag{4.1}$$

The generalized Friedmann constraint equation (2.18) now takes the simple form

$$\frac{\psi^2}{\epsilon^2}=1 \tag{4.2}$$

and may be employed to eliminate ψ. Since by definition $\epsilon \geqslant 0$, specifying one of the roots $\psi/\epsilon=\pm 1$ corresponds to choosing the sign of ψ. However, it follows from the definition in Eq. (2.19) that changing the sign of ψ is related to a time reversal of the dynamics. In what follows, we shall consider the case $\psi/\epsilon=+1$; the case $\psi/\epsilon=-1$ is qualitatively similar.

Introducing the new variables

$$\mu \equiv \frac{\sqrt{3}h}{\epsilon}, \quad \nu \equiv \frac{\rho}{2\epsilon^2}, \quad \lambda \equiv \frac{N}{\epsilon^2} \tag{4.3}$$

and a new dynamical variable

$$\frac{d}{dT} \equiv \frac{1}{\sqrt{3}\epsilon}\frac{d}{dt} \tag{4.4}$$

transforms Eqs. (2.14)–(2.17) to the three-dimensional autonomous system

$$\frac{d\mu}{dT}=\nu+\frac{\mu}{\sqrt{3}}[1-\mu^2-\lambda] \tag{4.5}$$

$$\frac{d\nu}{dT}=-2\nu\left[\mu+\frac{1}{\sqrt{3}}(\lambda+\mu^2)\right] \tag{4.6}$$

$$\frac{d\lambda}{dT}=\frac{2}{\sqrt{3}}\lambda(1-\mu^2-\lambda). \tag{4.7}$$

The phase space variables are bounded by the conditions $0 \leqslant \{\mu^2,\nu,\lambda\} \leqslant 1$ and also satisfy $\mu^2+\nu+\lambda \leqslant 1$. The sets $\nu=0$ and $\lambda=0$ are invariant sets corresponding to $\rho=0$ and $\dot{\beta}=0$, respectively. In addition, $\mu^2+\nu+\lambda=1$ is an invariant set corresponding to $\Lambda=0$. We note that the right-hand side of Eq. (4.7) is positive-definite and this simplifies the dynamics considerably.

The equilibrium points of the system (4.5)–(4.7) consist of the isolated equilibrium point

$$C: \mu=\nu=\lambda=0 \tag{4.8}$$

and the line of non-isolated equilibrium points

$$V: \nu=0,\ \lambda=1-\mu^2\ \ (\mu \text{ arbitrary}). \tag{4.9}$$

The eigenvalues associated with C are $\lambda_1=1/\sqrt{3}$, $\lambda_2=2/\sqrt{3}$ and $\lambda_3=0$. Although this isolated singular point C is non-hyperbolic, a simple analysis shows that it is a global source. The eigenvalues associated with V are

$$\lambda_1=-2\left(\mu+\frac{1}{\sqrt{3}}\right), \quad \lambda_2=-\frac{2}{\sqrt{3}} \tag{4.10}$$

and the third eigenvalue is zero since V is an equilibrium set. Therefore, on V the equilibrium points are saddles for $\mu \in [-1,-1/\sqrt{3})$ and local sinks for $\mu \in (-1/\sqrt{3},1]$. The phase portrait is given in Fig. 3.

It is also instructive to consider the dynamics on the boundary corresponding to $\lambda=0$, since the case $N=0$ is of physical interest in its own right as a four-dimensional model. In this case the ODEs reduce to the two-dimensional system

$$\frac{d\mu}{dT}=\nu+\frac{\mu}{\sqrt{3}}(1-\mu^2) \tag{4.11}$$

$$\frac{d\nu}{dT}=-2\mu\nu\left[1+\frac{1}{\sqrt{3}}\mu\right]. \tag{4.12}$$

The equilibrium points and their corresponding eigenvalues are

$$C: \ \mu=\nu=0; \ \lambda_1=\frac{1}{\sqrt{3}}, \quad \lambda_2=0 \tag{4.13}$$

$$S: \ \mu=-1, \ \nu=0; \ \lambda_1=-\frac{2}{\sqrt{3}}, \quad \lambda_2=2\left(1-\frac{1}{\sqrt{3}}\right) \tag{4.14}$$

$$A: \ \mu=1,\nu=0; \ \lambda_1=-\frac{2}{\sqrt{3}}, \ \lambda_2=-2\left(1+\frac{1}{\sqrt{3}}\right). \tag{4.15}$$

Point C is a non-hyperbolic equilibrium point; however, by changing to polar coordinates we find that C is a repeller with an invariant ray $\theta=\tan^{-1}(-\sqrt{3})$. The saddle S and the attractor A lie on the line V. The phase portrait is given in Fig. 4.

This concludes the analysis of the phase portraits for the FRW string cosmologies containing non-trivial NS-NS fields.

V. DISCUSSION

The phase portraits have a number of interesting features. In Fig. 1 the modulus field is frozen and the universe contracts from a singular initial state. The orbits in the vicinity of the equilibrium point R are asymptotic to the $p=p_-$ dilaton–vacuum solution (2.7). The axion is negligible and the kinetic energy of the dilaton dominates the energy-momentum tensor. As the collapse proceeds, however, the axion becomes dynamically more important and eventually induces a bounce. In the case of vanishing Λ, Eqs. (2.10) imply that the future attractor would correspond to the $p=p_+$ dilaton-vacuum solution. However, the combined effect of the axion and central charge is to cause the universe to evolve towards the equilibrium point A, where $\dot{\alpha} \to +\infty$, in

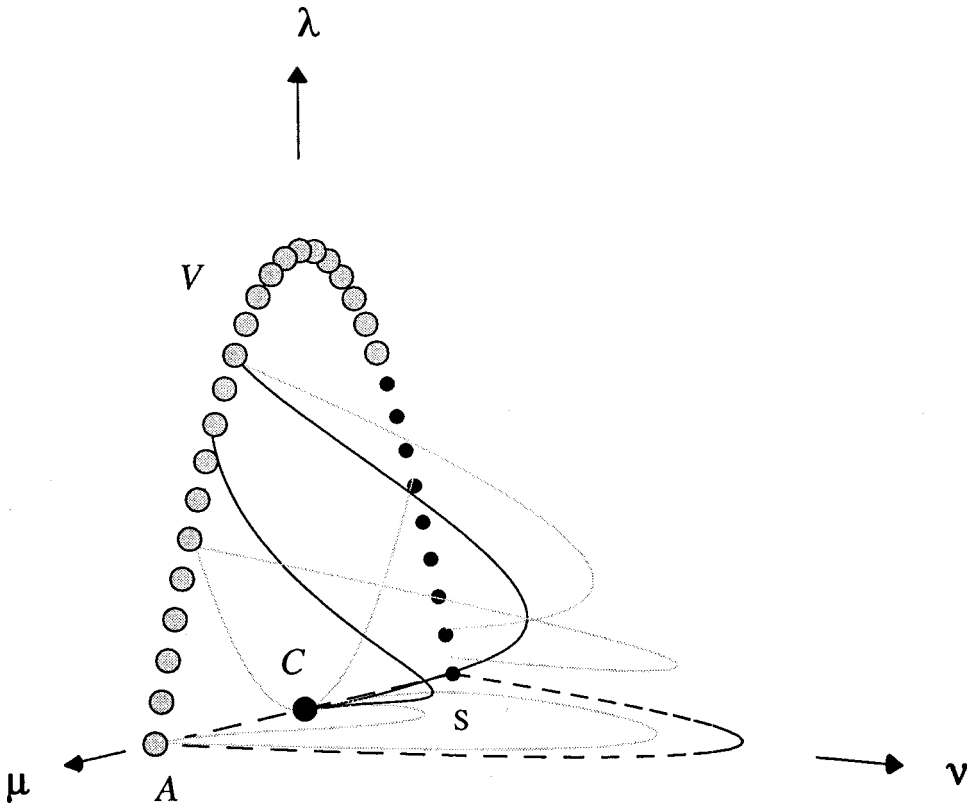

FIG. 3. The phase portrait of the system (4.5)–(4.7) corresponding to the ten-dimensional NS-NS model with positive central charge deficit ($\Lambda>0$). The root $\psi/\epsilon=+1$ of Eq. (4.2) has been chosen. Note that V denotes a line of non-isolated equilibrium points. See captions to Figs. 1 and 2.

a finite time. This behavior differs from that found when no axion field is present, because in this latter case there is no bounce [11,17].

This behavior can be understood by viewing the axion field as a membrane [21]. Since this field is constant on the surfaces of the homogeneity, the field strength of the two-form potential must be directly proportional to the volume form of the three-space. If the spatial topology of the universe is that of an isotropic three-torus, the axion field can be formally interpreted as a membrane wrapped around this torus [21]. As the universe collapses, the membrane resists being squashed into a singular point and this forces the universe to bounce into an expansionary phase. The cosmological constant then dominates the axion field as the latter's energy density decreases.

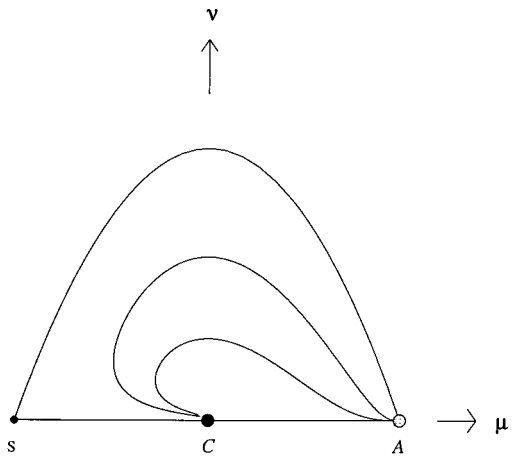

FIG. 4. Phase portrait of the system (4.11),(4.12) corresponding to the four-dimensional NS-NS model with positive central charge deficit ($\Lambda>0$). The root $\psi/\epsilon=+1$ of Eq. (4.2) has been chosen. See caption to Fig. 1.

The inclusion of a modulus field leads to a line of sources and sinks for the orbits (see Fig. 2). The axion field is dynamically negligible in the neighborhood of the equilibrium points. Moreover, a bouncing cosmology is no longer inevitable and there exist solutions that expand to infinity in a finite time. The solutions are asymptotic to the dilaton-moduli-vacuum solutions (2.9) near the lines $L_{\pm}$. The boundary points $\xi^2=1/3$ on these lines correspond to the limiting cases where $\dot{\alpha}^2=\dot{\beta}^2$. These represent the isotropic, ten-dimensional cosmology ($\dot{\alpha}=\dot{\beta}$) and its dual solution ($\dot{\alpha}=-\dot{\beta}$). In the latter solution, the ten-dimensional dilaton field, $\hat{\Phi}\equiv\Phi+6\beta$, is constant.

In Figs. 3 and 4, the isolated equilibrium point C corresponds to the ''linear dilaton vacuum'' solution (2.13) [11,18]. When the modulus is frozen, all trajectories evolve away from C towards the point A and approach the superinflationary $p=p_{-}$ dilaton-vacuum solution (2.7) defined over $t<0$. Some of the orbits evolving away from C represent contracting cosmologies and the effect of the axion is to

reverse the collapse in all these cases. For the rolling modulus solution (Fig. 3), the orbits tend to the dilaton-moduli-vacuum solutions as they approach the attractors (the sinks on V). As in the case of a negative central charge, the critical value $\mu^2=1/3$ corresponds to the case where $\dot{\alpha}^2=\dot{\beta}^2$. The other boundary of V is the point A representing the case where the kinetic energy of the modulus field vanishes. The qualitative behavior of models with $\psi<0$ is similar.

The results presented in this paper can be directly applied to the class of spatially flat and homogeneous Bianchi type I models by reinterpreting the physical meaning of the modulus field, β. The line element for the axisymmetric type I model is given by $ds^2=-dt^2+h_{ab}(t)dx^a dx^b$ $(a,b=1,2,3)$, where the metric on the surfaces of homogeneity is defined by $h_{ab}\equiv e^{2\alpha}\mathrm{diag}[-2\beta,\beta,\beta]$ [22]. The field equations derived from action (1.1) for this background are formally *identical* to those presented in Sec. II [23]. In this anisotropic model, however, the variables α and β now parametrize the averaged scale factor and the shear parameter of the universe, respectively. It would be interesting to investigate whether the above results can be employed to study the question of isotropization in string cosmologies.

Finally, we observe that all of the exact solutions corresponding to the equilibrium points of the governing autonomous systems of ODEs are self-similar since in each case the scale factor is a power-law function of cosmic time [see, for example, Eq. (2.8)] [24]. This result can be proved in general. For example, all of the equilibrium points of the system of ODEs (3.9)–(3.11) necessarily have $h=\mathrm{const}$ (leading to a power-law solution) or $\Lambda=0$, whence from the exact ''dilaton-moduli-axion'' solution (2.10) we can see that asymptotically the scale factor is power-law. This implies that exact self-similar solutions play an important role in determining the asymptotic behavior of string cosmologies [19].

In conclusion, therefore, we have presented a qualitative analysis of the spatially flat, FRW string cosmologies containing non-trivial dilaton, axion and modulus fields together with a central charge deficit (stringy cosmological constant). In all cases, variables were found that led to a compactification of the phase space and this allowed a complete stability analysis to be performed. Both four- and ten-dimensional models were studied by including the dynamics of the modulus field. The combined effects of the axion field and central charge deficit on the qualitative behavior of the dilaton-moduli-vacuum solutions (2.8) were determined. We found that such terms have a significant effect on the dynamics.

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