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ON THE CORRESPONDENCE BETWEEN EXACT SOLUTIONS IN KALUZA–KLEIN THEORY AND IN SCALAR–TENSOR THEORIES

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Using the formal equivalences between Kaluza–Klein gravity, Brans–Dicke theory and general relativity coupled to a massless scalar field, exact solutions obtained in one theory will correspond to analogous solutions in the other two theories. Often exact solutions in one theory are "rediscovered" since theory are not recognized as analogs of the corresponding solutions in one of the other theories. We review here a number of exact solutions in each of the theories, with an emphasis on identifying and presenting the higher-dimensional version of the solutions. We also briefly comment upon the formal equivalence between Kaluza–Klein theory and scalar–tensor theories in general.

1. Introduction

Recently there has been a resurgence of interest in the investigation of exact solutions of the five-dimensional Kaluza–Klein theory governed by the vacuum Einstein field equations and their relationship with the induced matter theory (Refs. 23, 68 and 69, see also references cited in these papers).

In addition, scalar–tensor theories of gravity have also been widely studied in recent years,[3,4,6,7,11,25,27,35] partially due to their relationship with the low energy limit of various unified field theories such as superstring theory[36]; in particular, the dimensional reduction of higher-dimensional gravity results in an effective scalar–tensor theory.[32,40]

For example, it is known that five-dimensional Kaluza–Klein theory with vacuum Einstein field equations (in which the metric is independent of the extra spatial dimension, ψ) is equivalent to the vacuum Brans–Dicke theory in four dimensions with the free parameter $\omega = 0$.[15] In particular, the five-dimensional Einstein–Hilbert action in the absence of matter is

$$S = \frac{1}{16\pi G_5} \int_{M_5} d^5x \sqrt{-{}^{(5)}g}\, {}^{(5)}R\,, \tag{1}$$

where ${}^{(5)}g$ is the determinant of the five-dimensional metric ${}^{(5)}g_{AB}$ $(A, B = 0, 1, 2, 3, 4)$ on the manifold $M_5 = M_4 \times M$ and ${}^{(5)}R$ is its associated Ricci scalar

(G_5 is the gravitational constant). If we write ($\alpha, \beta = 0,1,2,3$)

$$\begin{aligned} {}^{(5)}g_{\alpha\beta} &= \tilde{g}_{\alpha\beta} + \phi^2 A_\alpha A_\beta \,, \\ {}^{(5)}g_{\alpha 4} = {}^{(5)}g_{4\alpha} &= \phi^2 A_\alpha \,, \\ {}^{(5)}g_{44} &= \phi^2 \,, \end{aligned} \tag{2}$$

then if the fields are independent of the fifth coordinate $x^4 = \psi$ (the zero-mode, i.e. the effective low energy theory), we can integrate to obtain the four-dimensional action

$$S = \frac{1}{16\pi G}\int_{M_4} d^4x\sqrt{-\tilde{g}}\left[\phi\tilde{R} - \frac{1}{4}\phi^3\tilde{F}_{\alpha\beta}\tilde{F}^{\alpha\beta}\right] , \tag{3}$$

where $\tilde{R}$ is the Ricci scalar for the (four-dimensional) $\tilde{g}_{\alpha\beta}$ and $\tilde{F}_{\alpha\beta} \equiv A_{\alpha,\beta} - A_{\beta,\alpha}$ (and Newton's constant G is equal to G_5 integrated over ψ). It is always possible to locally choose coordinates in (2) such that ${}^{(5)}g_{4\alpha} = 0$ (i.e. $A_\alpha = 0$). With this coordinate choice, the action (3) is that of the vacuum Brans–Dicke theory (in four dimensions) with $\omega = 0$.

However, this formal equivalence is often not recognized when new solutions of either Kaluza–Klein theory or Brans–Dicke theory are found, and it is rarely utilized in obtaining new solutions. One exception is in the work of Romero and Tavakol[61] in which the formal equivalence of a cosmological Kaluza–Klein solution of Wesson and Ponce de Leon and the vacuum Brans–Dicke solution of O'Hanlon and Tupper (with $\omega = 0$) is noted.

This letter will repeatedly refer to three different theories of gravity, namely five-dimensional Kaluza–Klein theory, Brans–Dicke theory and general relativity coupled to a massless scalar field. Table 1 summarizes the notation and conventions used for the metric and scalar field in each theory, and also gives the reference for the relevant Einstein–Hilbert action.

Table 1. Notation used for each theory and the relevant Einstein–Hilbert action.

Theory	Metric and scalar field	Relevant action
Kaluza–Klein	${}^{(5)}g_{\alpha\beta} \to \tilde{g}_{\alpha\beta}, \phi$	Eqs. (1) and (3)
Brans–Dicke	$g_{\alpha\beta}, \Phi$	Eq. (8)
G.R.+M.S.F.	$\bar{g}_{\alpha\beta}, \bar{\phi}$	Eq. (12)

2. Static Spherically Symmetric Case

The general (asymptotically flat) three-parameter static, spherically symmetric vacuum solution in isotropic coordinates in the Brans–Dicke theory was given originally by Brans and Dicke[15] as

$$ds^2 = -\left(\frac{1-m/r}{1+m/r}\right)^{2/\alpha} dt^2 + \left(1+\frac{m}{r}\right)^4\left(\frac{1-m/r}{1+m/r}\right)^{\frac{2}{\alpha}(\alpha-\beta-1)}(dr^2 + r^2 d\Omega^2)\,, \tag{4}$$

and the static Brans–Dicke scalar field is given by

$$\Phi = \Phi_0 \left(\frac{1 - m/r}{1 + m/r} \right)^{\beta/\alpha} . \tag{5}$$

The constants Φ_0 and m are arbitrary, whereas α and β are constrained by

$$\alpha^2 = \left(\frac{\omega}{2} + 1 \right) \beta^2 + \beta + 1 . \tag{6}$$

Evidently, in the case that the Brans–Dicke parameter ω is zero, we recover the two-parameter (five-dimensional) Kaluza–Klein solution (with ${}^{(5)}g_{44} = \Phi^2 = \phi^2$, and the constant Φ_0 can be absorbed into the coordinate x^4) given by Gross and Perry[37] and Davidson and Owen.[28] The physical properties of this Kaluza–Klein solution was discussed by Davidson and Owen[28] and by Wesson.[67] The geometrical properties, also investigated in Refs. 12 and 70 have also been studied. However, it seems to have been overlooked that these properties are all known from earlier studies of Brans–Dicke solution (4), (5) (see, for example, Refs. 2 and 15).

The correspondence between Brans–Dicke theory and five-dimensional Kaluza–Klein theory can be generalized to $\omega \neq 0$ by the transformations[8]

$$\phi = \Phi^\mu , \qquad \tilde{g}_{\alpha\beta} = \Phi^{1-\mu} g_{\alpha\beta} , \tag{7}$$

(here and throughout, we define the constant $\mu \equiv \sqrt{1 + 2\omega/3}$, unless otherwise noted) which transform (3) into

$$S = \frac{1}{16\pi G} \int_{M_4} d^4x \sqrt{-g} \left[\Phi R - \frac{\omega}{\Phi} \nabla^\alpha \Phi \nabla_\alpha \Phi - \frac{1}{4} \Phi^{3\mu} F_{\alpha\beta} F^{\alpha\beta} \right] , \tag{8}$$

i.e. the Brans–Dicke theory with a vector potential (note that $F_{\alpha\beta} = \tilde{F}_{\alpha\beta}$, and ∇_α represents covariant differentiation with respect to the new metric $g_{\alpha\beta}$). Hence, any solution of (8) can be transformed into a Kaluza–Klein static solution with

$$\begin{aligned} {}^{(5)}g_{\alpha\beta} &= \Phi^{1-\mu} g_{\alpha\beta} + \Phi^{2\mu} A_\alpha A_\beta , \\ {}^{(5)}g_{\alpha 4} &= \Phi^{2\mu} A_\alpha , \\ {}^{(5)}g_{44} &= \Phi^{2\mu} . \end{aligned} \tag{9}$$

To illustrate, using (9), the Kaluza–Klein solution constructed from (4) and (5) is

$$\begin{aligned} ds^2 = &- \left(\frac{1 - m/r}{1 + m/r} \right)^{2/\alpha'} dt^2 + \left(1 + \frac{m}{r} \right)^4 \left(\frac{1 - m/r}{1 + m/r} \right)^{\frac{2}{\alpha'}(\alpha' - \beta' - 1)} (dr^2 + r^2 d\Omega^2) \\ &+ \left(\frac{1 - m/r}{1 + m/r} \right)^{2\beta'/\alpha'} d\psi^2 , \end{aligned} \tag{10}$$

where

$$\alpha' = \frac{\alpha}{1+\frac{1}{2}\beta-\frac{1}{2}\mu\beta}\,, \qquad \beta' = \frac{\beta\mu}{1+\frac{1}{2}\beta-\frac{1}{2}\beta\mu}\,,$$
$$\alpha = \frac{\alpha'\mu}{\mu-\frac{1}{2}\beta'+\frac{1}{2}\mu\beta'}\,, \qquad \beta = \frac{\beta'}{\mu-\frac{1}{2}\beta'+\frac{1}{2}\beta'\mu}\,.$$

Substituting (α', β') for (α, β), Eq. (6) becomes $\alpha'^2 = \beta'^2 + \beta' + 1$ which is the consistency relationship for the known Kaluza–Klein static spherically symmetric solution.

3. Axial Symmetry

Brans–Dicke theory is formally equivalent to general relativity plus a massless scalar field[17,29,48,65] under the field redefinition and conformal transformation (respectively)

$$\begin{aligned} \bar{\phi} &= \sqrt{3}\mu \ln \Phi\,, \\ \bar{g}_{\alpha\beta} &= \Phi g_{\alpha\beta}\,, \end{aligned} \tag{11}$$

from which (8) becomes

$$S = \frac{1}{16\pi G}\int_{M_4} d^4x\sqrt{-\bar{g}}\left[\bar{R} - \frac{1}{2}\bar{\nabla}^\alpha\bar{\phi}\bar{\nabla}_\alpha\bar{\phi} - \frac{1}{4}e^{\sqrt{3}\bar{\phi}}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta}\right] \tag{12}$$

($\bar{F}_{\alpha\beta} \equiv F_{\alpha\beta}$ and $\bar{F}^{\alpha\beta}$ is obtained from $\bar{g}^{\alpha\beta}$). Therefore Kaluza–Klein solutions can be obtained from known scalar field solutions in general relativity as well. For clarity, let us explicitly mention the field equations for some of the aforementioned theories. Gravity as described by (8) with $A_\alpha = 0$ is governed by the field equations

$$R_{\alpha\beta} = \frac{\nabla_\alpha\nabla_\beta\Phi}{\Phi} + \frac{\omega}{\Phi^2}\nabla_\alpha\Phi\nabla_\beta\Phi\,,$$
$$\Box\Phi = 0$$

(where $\Box$ is the d'Alembertian), whereas the conformal transformation to $(\bar{\phi}, \bar{g}_{\alpha\beta})$ changes these equations to

$$\bar{R}_{\alpha\beta} = \frac{1}{2}\bar{\nabla}_\alpha\bar{\phi}\bar{\nabla}_\beta\bar{\phi}\,,$$
$$\bar{\Box}\bar{\phi} = 0\,.$$

However, the latter is often expressed in slightly different forms in the literature [e.g. in Ref. 2, $\bar{R}_{\alpha\beta} = -2\bar{\nabla}_\alpha\bar{\varphi}\bar{\nabla}_\beta\bar{\varphi}$ was used].

There are several examples of static, axially symmetric vacuum (Kerr-like) solutions both in Brans–Dicke theory and in theories where general relativity is coupled to a massless scalar field. This suggests that Kerr-like solutions in Kaluza–Klein theory can also be obtained. Krori and Bhattacharjee[42] have found Kerr-like solutions in Brans–Dicke theory and have extended these to Demianski-type solutions,[43] whereas Agnese and La Camera[2] have obtained Kerr-like solutions in general relativity theory with a scalar field. In comparing the Kerr-like solutions between the two sets of data, the transformation (11) can be used to verify that the two metrics are indeed equivalent, while the scalar fields in each theory are *only* equivalent in the absence of the rotational parameter. The Kaluza–Klein Kerr-like solutions corresponding to either theory can be written as

$$\begin{aligned} ds^2 = \frac{1}{\xi}\Bigg\{ & C^f dt^2 - 2a\sin^2\theta(C^f - 1)dt\,d\varphi - (r^2 + a^2\cos^2\theta)C^{1-f}\left(\frac{dr^2}{\Delta} + d\theta^2\right) \\ & - \sin^2\theta\left[(r^2 + a^2\cos^2\theta)C^{1-f} + a^2\sin^2\theta(2 - C^f)\right]d\varphi^2\Bigg\} - \xi^2 d\psi^2\,, \end{aligned} \tag{13}$$

where

$$C = 1 - \frac{2\eta r}{r^2 + a^2\cos^2\theta}\,, \tag{14}$$

$$\Delta = r^2 + a^2 - 2\eta r\,. \tag{15}$$

For the Kerr-like solution of Ref. 2 one has $f = m/\eta$, $\eta \equiv \sqrt{m^2 + \sigma^2}$ and

$$\xi = \left(\frac{r - \eta - \sqrt{\eta^2 - a^2}}{r - \eta + \sqrt{\eta^2 - a^2}}\right)^{\sigma/(2\sqrt{3}\sqrt{\eta^2 - a^2})}\,. \tag{16}$$

For the Kerr-like solution of Ref. 42 one has $f = 1/\lambda + c/(2\lambda)$ and

$$\xi = C^{c\mu/(2\lambda)}\,. \tag{17}$$

Analogously, Krori and Bhattacharjee's Demianski-like solution[43] (of which their Kerr-like solution is a subclass) can be expressed in a five-dimensional formalism. We note, however, that we have not been able to confirm any of these as solutions, despite the fact that corrections have been made to Agnese and La Camera's solution.[1]

There is a class of axially symmetric solutions already known in Kaluza–Klein theory by Bruckman.[16] This appears to be a larger class of solutions than the Brans–Dicke solutions of Krori *et al.*[43] or the relativistic scalar field solutions of Agnese *et al.*,[2] since Bruckman's solutions contain prolate and oblate configurations and do not necessarily reduce to spherical symmetry when there is no rotation. However, the constants in these solutions can be adjusted so that one may obtain spherical symmetry in such a limit. Ma[46] used Bruckman's solutions in this way to produce Brans–Dicke Kerr-like solutions, although it is not apparent that these solutions are equivalent to the aforementioned axially symmetric solutions except in two limits; one limit is when the five-dimensional metrics reduce to a four-dimensional Kerr solution plus a flat fifth dimension ($g_{44} = 1$), and the second limit is when the rotation parameter is set to zero and the solutions reduce to the known Brans–Dicke spherically symmetric class of solutions (Eqs. (4) and (5)).

Myers and Perry[55] also obtained Kerr-like solutions, but for arbitrary dimensions. Their five-dimensional solution differs from Bruckman's in that it contains neither the 4-D Kerr solution (with $g_{44} = 1$) nor the known 5-D spherically-symmetric solution as special cases. This solution give rise to the following Brans–Dicke solution (which apparently has not been written down):

$$\begin{aligned} ds^2 = (r\cos\theta)^{1-1/\mu}\Bigg\{ &-\left(1 - \frac{M}{r^2 + a^2\cos^2\theta}\right)dt^2 + 2\frac{aM\sin^2\theta}{r^2+a^2\cos^2\theta}dt\,d\varphi \\ &+ (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2+a^2-M} + d\theta^2\right) \\ &+ \sin^2\theta\left[(r^2+a^2) + \frac{a^2M\sin^2\theta}{r^2+a^2\cos^2\theta}\right]d\varphi^2\Bigg\} \end{aligned} \tag{18}$$

and

$$\Phi = (r\cos\theta)^{1/\mu}\,. \tag{19}$$

4. Solutions with $A_\alpha \neq 0$

We turn our attention next to solutions in which the five-dimensional solution has $A_\alpha \neq 0$ [see Eq. (9)], or, equivalently, solutions in the Brans–Dicke theory which contains a vector field. First, there is the Kaluza–Klein static, spherically symmetric vacuum solution of Liu and Wesson[45]

$$\begin{aligned} ds^2 = &\frac{(1-k)B^a}{1-kB^{a-b}}dt^2 - B^{-a-b}dr^2 - r^2B^{1-a-b}\,d\Omega^2 \\ &- \frac{B^b - kB^a}{1-k}(dy + A_\gamma dx^\gamma)\,, \qquad \gamma = 0,1,2,3 \end{aligned} \tag{20}$$

in which the vector potential is given by

$$A_\alpha = \left[\frac{-\sqrt{k}(1-B^{a-b})}{1-kB^{a-b}}, 0, 0, 0\right]\,, \tag{21}$$

where

$$B = \left(1 - \frac{2(1-k)M}{r}\right)\,, \tag{22}$$

and k, a and b are constants satisfying $a^2 + ab + b^2 = 1$. (When $k = 0$, the known five-dimensional static, spherically symmetric solution is recovered.) Duff[30] has also discussed a five-dimensional solution with $A_\alpha \neq 0$ corresponding to (20)–(22) with $a = 1$, $b = 0$ (this is realized by translating Liu and Wesson's origin by $r \to r - 2Mk$ and setting Duff's constants r_+ and r_- to $2M$ and $2Mk$, respectively). Duff[30] also discussed a magnetic monopole solution with either $A_\alpha = [0, 0, \sqrt{r_+ r_-}\phi \sin\theta + \chi(\theta), 0]$ (where $\chi(\theta)$ is arbitrary) or $A_\alpha = [0, 0, 0, \sqrt{r_+ r_-}\cos\theta]$. The latter solution can be reduced to the magnetic monopole solution of Gross and Perry.[37] In Gross and Perry[37] there are several examples of five-dimensional solutions with $A_\alpha \neq 0$, and we simply note them here with the corresponding equations from their paper: the magnetic monopole solution [Eq. (18)], the non-interacting multi-monopole solution [Eqs. (26) and (27)], the Kaluza–Klein dipole solution [Eq. (30)] and the single dipole–single monopole combination [Eq. (35)]. All of these Kaluza–Klein solutions have analogs in the Brans–Dicke theory.

In addition, Agnese and La Camera[2] obtained a Reissner–Nordstrøm-like space–time coupled to a massless scalar field, the vector field of which is $A_\alpha = (V(r), 0, 0, 0)$ where

$$|g_{\theta\theta}| \frac{d}{dr} V(r) = q\,, \tag{23}$$

defining q as the total electric charge of the body. Finally, García and Mitskiévić[33] obtained two sets of solutions in the Brans–Dicke formalism coupled to a vector field. The first set is a class of Kerr–Newman metrics with a scalar field defined by an infinite sum of (asymptotically flat) Legendre polynomials. When all the constants in the Legendre polynomials vanish, a Kerr–Newman (charged) metric with a constant scalar field is recovered. The second set of solutions is a class of Brans–Dicke solutions analogous to the charged Tomimatsu–Sato solution (see Ernst[31]).

5. Self-Similar Solutions

We now turn our attention to exact self-similar solutions. Self-similar solutions of the first kind are solutions admitting a homothetic vector; that is, there exists a vector field, ξ, such that

$$\mathcal{L}_\xi g_{ab} = 2c g_{ab}\,, \tag{24}$$

where $\mathcal{L}_\xi$ is the Lie derivative along ξ and c is a constant. In the event that $c = 0$, ξ is then a Killing vector. When $c \neq 0$, i.e. ξ is a proper homothetic vector, units can be chosen to set $c = 1$. Self-similarity in cosmology and generalized self-similarity have recently been reviewed in Carr and Coley[18] and in Coley[24] (see also references cited therein). Self-similar solutions also exist in scalar–tensor and higher-dimensional gravity theories. For example, in four dimensions the spherically symmetric metric of the form

$$ds^2 = -e^{F(r/t)}dt^2 + e^{G(r/t)}dr^2 + r^2 e^{H(r/t)}d\Omega^2 \tag{25}$$

admits the homothetic vector

$$\xi^a = (t,\, r,\, 0,\, 0)\,. \tag{26}$$

Note that this metric remains invariant under the rescalings $r \to ar$ and $t \to at$ (a constant).

In order to study the possible equivalence between self-similar solutions in Kaluza–Klein theory, Brans–Dicke theory and general relativity coupled to a massless scalar field, let us obtain the five-dimensional generalization of (26). If we assume that $\xi^a = (t, r, 0, 0, 0)$, then the corresponding metric that satisfies (24) is

$$ds^2 = -e^{\tilde{F}(r/t)}dt^2 + e^{\tilde{G}(r/t)}dr^2 + r^2\left(e^{\tilde{H}(r/t)}d\Omega^2 + e^{\tilde{L}(r/t)}dy^2\right)\,. \tag{27}$$

Alternatively, the metric

$$ds^2 = -e^{\tilde{F}(r/t)}dt^2 + e^{\tilde{G}(r/t)}dr^2 + r^2e^{\tilde{H}(r/t)}d\Omega^2 + e^{\tilde{L}(r/t)}dy^2 \tag{28}$$

admits the homothetic vector $\xi^a = (t, r, 0, 0, y)$. Hence, given a scalar field solution (either in Brans–Dicke theory or relativistic massless scalar field theory), it is the form of the scalar field which will determine the form of the five-vector ξ since it is the scalar field which determines the form of the dy^2 term in the five-dimensional metric.

There is an extensive literature of self-similar solutions in general relativity.[18] However, let us restrict our attention here to spherically symmetric solutions in either Brans–Dicke theory or general relativity coupled to a scalar field, since these solutions are important in the study of naked singularities and critical phenomena.[18] Roberts,[60] in an attempt to find counterexamples to the cosmic censorship hypothesis, found an analytic self-similar, spherically symmetric solution which contained a naked singularity. This solution was "generalized" by de Oliveria[56] in an investigation of critical phenomena which was instigated by the numerical study of the spheical symmetric gravitational collapse of a massless scalar field.[22]

In most of the aforementioned works the line element is expressed in double-null coordinates (see, for instance, Refs. 13 and 14)

$$ds^2 = -e^{2\sigma(u,v)}dudv + f(u,v)d\Omega^2\,. \tag{29}$$

The most general solution to the governing field equations is given by

$$\begin{gathered} e^{2\sigma(u,v)} = 1\,, \\ f(u,v) = \frac{(1 + C_y^2 - C_q^2 \pm 2C_y)u^2}{4a^2(1 + C_q^2 - C_y^2)} - \frac{uv}{2} + \frac{a^2(1 + C_y^2 - C_q^2 \mp 2C_y)v^2}{4(1 + C_q^2 - C_y^2)}\,, \end{gathered} \tag{30}$$

where the scalar field is given by

$$\bar{\phi} = \pm \ln\left[\frac{(C_y - C_q \pm 1)u + a^2(C_y - C_q \mp 1)v}{(C_y + C_q \pm 1)u + a^2(C_y + C_q \mp 1)v}\right] \tag{31}$$

(C_y, C_q and a are all constants). This is Roberts' solution when $C_q = -C_y$, and also the solution of de Oliveira[56] and of de Oliveira and Cheb-Terrab[57] under redefinitions of the constants C_q and C_y. By "most general" above, we mean that with the assumption of self-similarity, the field equations can be directly integrated with (30) and (31) as the solution. Although many authors initially assume e^σ = constant, it can be shown that σ *must* be constant in order for the field equations to be satisfied and can be arbitrarily set to zero without loss of generality.

Using the coordinate transformations $u = a(t + r)$ and $v = \frac{1}{a}(t - r)$, and applying (11) and (7), the corresponding five-dimensional solution is

$$ds^2 = \zeta^{\mp\frac{1}{\sqrt{3}}}\left(-dt^2 + dr^2 + r^2\,\frac{1 \pm C_y\frac{t}{r} + (C_y^2 - C_q^2)\frac{t^2}{r^2}}{1 + C_q^2 - C_y^2}\,d\Omega^2\right) + \zeta^{\pm\frac{2}{\sqrt{3}}}dy^2\,, \tag{32}$$

where

$$\zeta = 1 - \frac{2C_q}{C_y + C_q \pm t/r}\,. \tag{33}$$

Note that here ${}^{(5)}g_{yy} = {}^{(5)}g_{yy}(r/t)$ and so the corresponding five-dimensional homothetic vector must be $\xi^a = (t, r, 0, 0, y)$.

Most solutions based on Roberts' solution are found within the framework of general relativity coupled to a massless scalar field. However, Chiba and Soda[19] and de Oliveira[56] have attempted to find a corresponding solution in the Brans–Dicke formalism, by using the relations (11). Therefore, the corresponding Kaluza–Klein solution obtained from these Brans–Dicke solutions will be the same as (32).

There are other examples of self-similar, spherically-symmetric solutions. In particular, Soda and Hirata[63] studied $(4 + D)$-dimensional gravity coupled to a massless scalar field. However, since their $(4 + D)$-dimensional energy-momentum tensor is not that of a vacuum, the conformal transformations in this letter do not necessarily hold. Also, Wang and de Oliveira[66] have found a solution in which the metric functions and massless scalar field involve Heaviside functions, which was obtained by matching two different solutions along a null hypersurface. Again, one is then able to produce analogous solutions both in the Brans–Dicke framework as well as the Kaluza–Klein framework.

6. Concluding Remarks

In this letter we have noted the formal equivalence of five-dimensional vacuum general relativity, Brans–Dicke theory and general relativity plus a massless scalar field, and we have utilized this equivalence to obtain analog exact solutions. Often such analogs are not recognized as such and apparently known solutions are rediscovered. We are particularly interested in recognizing and obtaining exact solutions of the five-dimensional Kaluza–Klein theory governed by the vacuum Einstein field equations.

The generation of exact solutions can be done, and is indeed done, by utilizing the formal dynamical equivalence (via conformal transformations, field redefinitions and dimensional reduction) of higher-dimensional (≥ 5) Kaluza–Klein theories, Brans–Dicke theory and generalized scalar–tensor theories (cf. Refs. 10 and 64), general relativity and a number of (coupled) scalar fields, and higher derivative (e.g. $f(R)$) theories, at least in the classical sectors of these theories.[5,9,11,25,39–41,44,48–53,59,62,65,71]

For example, Freund[32] and Holman *et al.*[41] have shown that a $(4 + D)$-dimensional theory with a certain ansatz is equivalent to Brans–Dicke theory with a specific value for the coupling parameter, namely $\omega = 1/D - 1$. A conformal transformation similar to (9) (with $A_\alpha = 0$) can be made to generalize this correspondence to arbitrary ω (via $\phi = \Phi^{\mu/D}$ and $\tilde{g}_{\alpha\beta} = \Phi^{1-\mu} g_{\alpha\beta}$ where now $\mu \equiv \sqrt{(\omega + 3/2)/(1/D + 1/2)}$).

In addition, it has been shown that the field equations of the $(2n + 4)$th order gravity, derived from a Lagrangian that depends on R, $\Box R$ and $\Box^n R$ (where $\Box$ is the d'Alembertian), are equivalent to the field equations of Brans–Dicke gravity (with $\omega = 0$) with an interaction potential for the Brans–Dicke field and n further scalar fields.[48,65]

In particular, the Kerr-like solutions of Myers and Perry[55] in arbitrary dimensions will give rise to analogous four-dimensional scalar–tensor solutions, as will the various higher-dimensional static spherically symmetric exact solutions and solutions that admit a D-dimensional space of constant curvature.[9,21,23,38,47,58,59,72] A particular example of a $(4 + D)$-dimensional manifold is the 1354-dimensional zero-curvature Kasner solution, where the metric is given by

$$ds^2 = -dt^2 + t^{\frac{142}{123}}(dr^2 + r^2 d\Omega^2) + t^{-\frac{2}{1845}} d\,\Xi^2\,, \tag{34}$$

and where $d\,\Xi^2 \equiv \sum_{i=4}^{1353} (dy^i)^2$.

The plethora of exact cosmological solutions of Brans–Dicke theory and its generalized scalar–tensor theories[3,4,6,7,44,54] will give rise to a host of analogous higher-dimensional exact solutions. For example, if one performs the conformal transformation $\tilde{g}_{\alpha\beta} = \Phi e^{-f(\Phi)} g_{\alpha\beta}$ and the field redefinition $\phi = e^{f(\Phi)}$, where $f(\Phi) = \int \sqrt{1 + \frac{2}{3}\omega(\Phi)} \frac{d\Phi}{\Phi}$, then one is able to transform between a five-dimensional Kaluza–Klein solution and a scalar–tensor theory in which $\omega = \omega(\Phi)$.

Finally, it is important to note that the formal equivalences between Brans–Dicke theory, Kaluza–Klein theory and relativistic scalar field theory (as reviewed in this letter) is only valid in the *absence of matter fields*. In general, as soon as matter fields are introduced, the scalar fields will interact with the matter in a nontrivial manner and the formal equivalences between these theories may be broken (see Refs. 45, 65 and references therein).

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